# NOTE ON THE DISTORTION OF $(2, q)$-TORUS KNOTS

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Abstract. We show that the distortion of the $(2, q)$-torus knot is not bounded linearly from below.

## 1. Introduction

The notion of distortion was introduced by Gromov [1]. If $\gamma$ is a rectifiable simple closed curve in $\mathbf{R}^3$, then its distortion $\delta$ is defined as

$$\delta(\gamma) = \sup_{v,w\in\gamma} \frac{d_\gamma(v, w)}{|v - w|},$$

where $d_\gamma(v, w)$ denotes the length of the shorter arc connecting $v$ and $w$ in $\gamma$ and $|\cdot|$ denotes the euclidean norm on $\mathbf{R}^3$. For a knot $K$, its distortion $\delta(K)$ is defined as the infimum of $\delta(\gamma)$ over all rectifiable curves $\gamma$ in the isotopy class $K$. Gromov [3] asked in 1983 if every knot $K$ has distortion $\delta(K) \leq 100$. The question was open for almost three decades until Pardon gave a negative answer. His work [4] presents a lower bound for the distortion of simple closed curves on closed PL embedded surfaces with positive genus. Pardon showed that the minimal intersection number of such a curve with essential discs of the corresponding surface bounds the distortion of the curve from below. In particular for the $(p, q)$-torus knot he obtained the following bound.

**Theorem** ([4]). *Let $T_{p,q}$ denote the $(p, q)$-torus knot. Then*

$$\delta(T_{p,q}) \geq \frac{1}{160} \min(p, q).$$

By considering a standard embedding of $T_{p,p+1}$ on a torus of revolution one obtains $\delta(T_{p,p+1}) \leq const \cdot p$, hence for $q = p + 1$ Pardons result is sharp up to constants.

An alternative proof for the existence of families with unbounded distortion was given by Gromov and Guth [2]. In both works the answer of Gromovs question was obtained by an estimate of the conformal length, which is up to a constant a lower bound for the distortion of rectifiable closed curves. However the conformal length is in general not a good estimate for the distortion. For example one finds easily

an embedding of the $(2, q)$-torus knot with conformal length $\leq 100$ and distortion $\geq q$ by looking at standard embeddings on a torus of revolution with suitable dimensions. In particular neither Pardon's nor Gromov and Guth's arguments yield lower bounds for $\delta(T_{2,q})$. While Pardon writes that surely $\lim_{q\to\infty} \delta(T_{2,q}) = \infty$ and that there are to his knowledge no known embeddings of $T_{2,q}$ with sublinear distortion [4] [p.2], Gromov and Guth [2] write that the distortion of $T_{2,q}$ appears to be $q$ up to constants [p.33]. In this article we show that the growth rate of $\delta(T_{2,q})$ is in fact sublinear in $q$.

**Theorem 1.** *Let $q \geq 50$. Then $\delta(T_{2,q}) \leq 7q/\log q$. In particular the distortion of the $(2, q)$-torus knot is not bounded linearly from below.*


## 2. Acknowledgments

Most thank is owed to Sebastian Baader and Pierre Dehornoy for their inspiring introduction to the topic. I would like to thank John Pardon for useful comments on the presented example and Peter Feller, Filip Misev, Johannes Josi and Livio Liechti for many mathematical discussions. I also thank Paul Frischknecht for the pictures.


## 3. Proof

In order to prove Theorem 1 we need to give for every odd integer $q \geq 50$ an embedding $\gamma$ of the $(2, q)$-torus knot with distortion smaller or equal to $7q/\log q$. The idea is to use a logarithmic spiral. Let $S$ be a logarithmic spiral of unit length starting at its center $0 \in \mathbf{R}^3$ and ending at some $u \in \mathbf{R}^3$. An elementary calculation shows that its distortion is equal to $1/|u|$. For another path $\alpha \subset \mathbf{R}^3$ of unit length and diameter $\leq 2|u|$ with endpoints $\{v, w\} = \partial\alpha$ we get

$$\delta(\alpha) \geq \frac{d_\alpha(v, w)}{|v - w|} = \frac{1}{|v - w|} \geq \frac{1}{2|u|} = \frac{\delta(S)}{2}.$$

Hence up to at most a factor 2 the logarithmic spiral has the smallest distortion among all paths for a prescribed pathlength-pathdiameter-ratio. It seems therefore natural to pack the $q$ windings of the $(2, q)$-torus knot into a logarithmic spiral in order to minimize distortion.

*Proof of Theorem 1.* Let $q$ be an odd integer greater or equal to 50, and $k = \log(q)/2\pi q$. We define the embedding $\gamma$ as the union of a segment of the logarithmic spiral with slope $k$, denoted by $S$, and a piecewise linear part, denoted by $L$, see Figure 1. The segment of the logarithmic spiral S is contained in the yellow painted $(x, z)$ plane and

parametrized by

$$\varphi : [0, \pi q] \to \mathbf{R}^2, \quad \varphi(s) = e^{ks} \cdot \begin{pmatrix} \cos(s) \\ \sin(s) \end{pmatrix},$$

see Figures 1 and 2. The segment of the piecewise linear part L is in the green painted $(x, y)$ plane, see Figures 1 and 3. Note that

$$|\varphi(\pi q)| = e^{k\pi q} = \sqrt{q} \quad \text{and} \quad |\varphi(0)| = 1,$$

hence the lengths defining $L$ in Figure 3 are chosen such that the union $\gamma$ of $S$ and $L$ is the simple closed curve illustrated in Figure 1. The linear segments $L_1$ and $L_2$ indicated in Figure 3 are named because of their special role in the following computations.

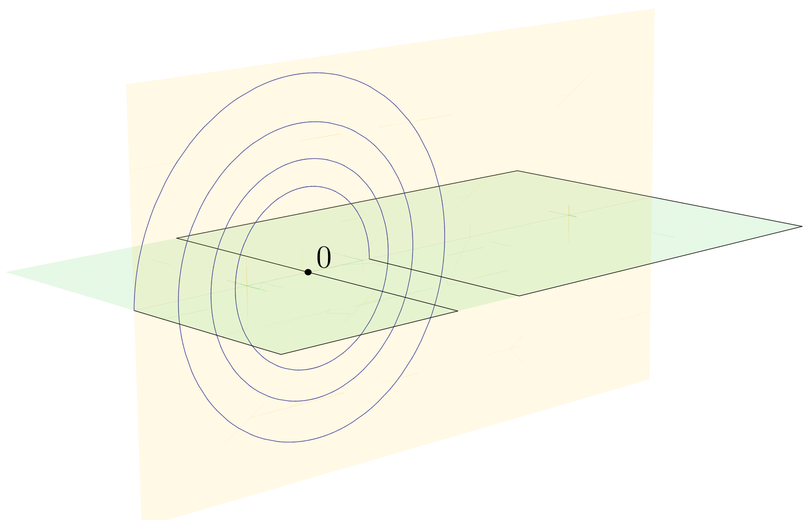


FIGURE 1. The embedding $\gamma$ for $q = 7$.

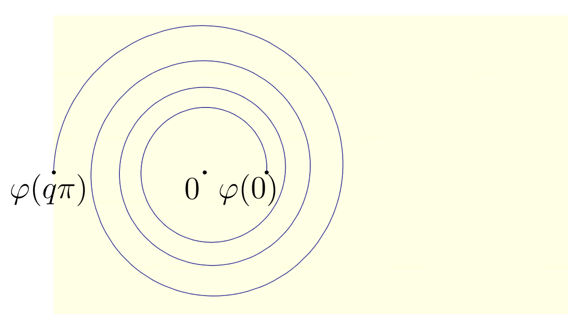


FIGURE 2. The logarithmic spiral S in the $(x, z)$ plane.

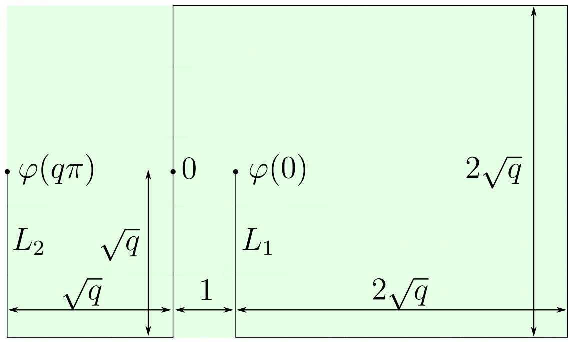


FIGURE 3. The linear part L in the $(x, y)$ plane.

To see that the obtained curve is an embedded $(2, q)$-torus knot, we perturb $\gamma$, see Figure 4. This simple closed curve is ambient isotopic in $\mathbf{R}^3$ to $\gamma$ and if we project it onto the $(x, y)$ plane, we see a well known diagram of the $(2, q)$-torus knot, see Figure 5.

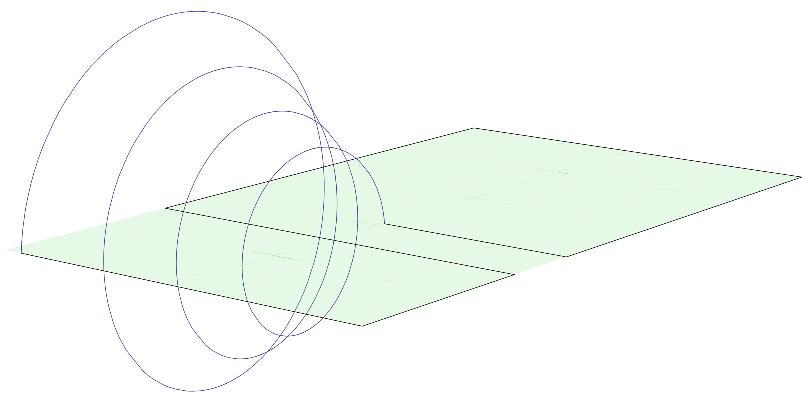

FIGURE 4. Perturbation of $\gamma$.

We now estimate the distortion of $\gamma$. One has to show that

$$\frac{d_\gamma(v,w)}{|v-w|} \leq \frac{7q}{\log q}$$

for all pairs of points $v, w \in \gamma$. A calculation shows that

$$\frac{1}{k} \cdot \sqrt{2k^2+1} = \frac{2\pi q}{\log q} \cdot \sqrt{2(\log q/2\pi q)^2+1} \leq \frac{7q}{\log q}$$

for all positive integers. Therefore, it suffices to show that

$$\frac{d_\gamma(v,w)}{|v-w|} \leq \frac{\sqrt{2k^2+1}}{k}.$$

In order to do this, we distinguish four cases.

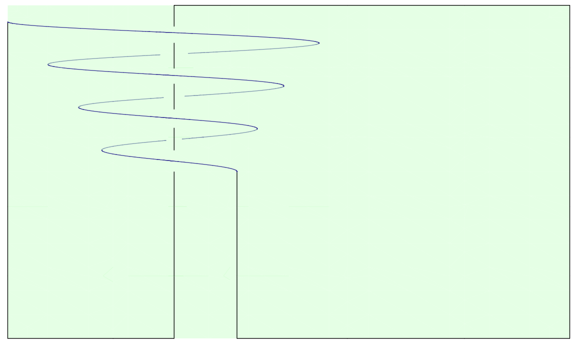

FIGURE 5. Projection onto the $(x, y)$ plane.

*Case 1:* $v, w \in S$. Let $0 \leq s \leq t \leq \pi q,\ v = \varphi(s), w = \varphi(t)$. From

$$|\varphi'(r)| = \left| \begin{pmatrix} \cos(r) & -\sin(r) \\ \sin(r) & \cos(r) \end{pmatrix} \begin{pmatrix} ke^{kr} \\ e^{kr} \end{pmatrix} \right| = \left| \begin{pmatrix} ke^{kr} \\ e^{kr} \end{pmatrix} \right| = \sqrt{k^2+1} \cdot e^{kr},$$

we get

$$\begin{aligned} d_\gamma(v, w) &\leq d_S(v, w) \\ &= \int_s^t |\varphi'(r)| dr \\ &= \sqrt{k^2+1} \int_s^t e^{kr} dr \\ &= \tfrac{\sqrt{k^2+1}}{k} \cdot (e^{kt} - e^{ks}) \\ &= \tfrac{\sqrt{k^2+1}}{k} \cdot (|\varphi(t)| - |\varphi(s)|) \\ &= \tfrac{\sqrt{k^2+1}}{k} \cdot (|w| - |v|). \end{aligned}$$

Since $|w - v| \geq |w| - |v|$, we conclude that

$$\frac{d_\gamma(v, w)}{|v - w|} \leq \frac{\sqrt{k^2+1}}{k} \cdot \frac{(|w| - |v|)}{(|w| - |v|)} = \frac{\sqrt{k^2+1}}{k}.$$

*Case 2:* $v \in L_1 \cup L_2,\ w \in S$. We consider the case where $v \in L_1$. The idea is to find the maximum of

$$\frac{d_\gamma(v, w)}{|v - w|}$$

for fixed $w$ and varying $v$. Let $t = |v - \varphi(0)|$, $a = |\varphi(0) - w|$, and $b = d_S(\varphi(0), w)$, see Figure 6.

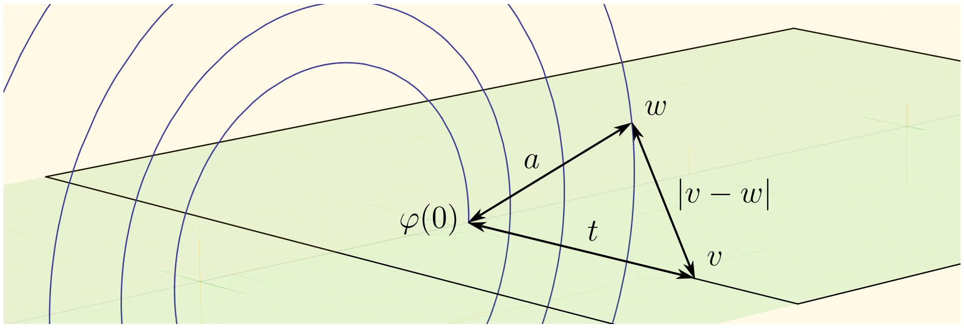


FIGURE 6.

Note that

$$|v - w| = \sqrt{t^2 + a^2}$$

and

$$d_\gamma(v, \varphi(0)) = |v - \varphi(0)| = t.$$

We get

$$\frac{d_\gamma(v,w)}{|v-w|} \leq \frac{d_\gamma(v,\varphi(0)) + d_S(\varphi(0),w)}{|v-w|} = \frac{t+b}{\sqrt{t^2+a^2}} =: f(t).$$

Deriving $f$ with respect to $t$ yields a unique critical point at $t = a^2/b$:

$$0 = f'(t) = \frac{a^2 - bt}{(a^2+t^2)^{3/2}} \qquad \Longleftrightarrow \qquad t = a^2/b.$$

Since $a^2/b$ is the only critical point, $f(\infty) = 1 \leq b/a = f(0)$ and

$$f(0) = \frac{b}{a} \leq \frac{\sqrt{a^2+b^2}}{a} = \frac{\frac{a^2}{b}+b}{\sqrt{(\frac{a^2}{b})^2+a^2}} = f(a^2/b),$$

$a^2/b$ must be a global maximum. Consequently we get

$$
\begin{aligned}
\frac{d_\gamma(v,w)}{|v-w|} &\leq \frac{\sqrt{a^2+b^2}}{a} \\
&= \sqrt{1+\left(\frac{b}{a}\right)^2} \\
&= \sqrt{1+\left(\frac{d_S(\varphi(0),w)}{|\varphi(0)-w|}\right)^2} \\
&\overset{\text{Case1}}{\leq} \sqrt{1+\left(\tfrac{\sqrt{k^2+1}}{k}\right)^2} \\
&= \frac{\sqrt{2k^2+1}}{k}.
\end{aligned}
$$

In the case where $v \in L_2$, we make the estimate with the path that connects $v$ with $w$ through $\varphi(\pi q)$. It works exactly the same and yields the same estimate.

*Case 3:* $v, w \in L$. Consider Figure 3 and note that all pairs of points $v, w \in L$ that could cause big distortion are of euclidean distance at least 1. Therefore we get

$$\frac{d_\gamma(v,w)}{|v-w|} \leq l(L) = 11\sqrt{q}+1.$$

A calculation shows that

$$11\sqrt{q}+1 \leq \frac{2\pi q}{\log q} = \frac{1}{k}$$

for q greater or equal to 50.

*Case 4:* $v \in L \setminus (L_1 \cup L_2), w \in S$. Note that for these pairs of points we have

$$|v-w| \geq |w|.$$

We estimate $d_\gamma(v,w)$ using results of Case 1 and 3:

$$
\begin{aligned}
d_\gamma(v,w) &\leq d_L(v,\varphi(0)) + d_S(\varphi(0),w) \\
&\leq \tfrac{1}{k} + \tfrac{\sqrt{k^2+1}}{k} \cdot (|w|-1) \\
&\leq \tfrac{\sqrt{k^2+1}}{k} \cdot |w|.
\end{aligned}
$$

We conclude that

$$\frac{d_\gamma(v,w)}{|v-w|} \leq \frac{\frac{\sqrt{k^2+1}}{k} \cdot |w|}{|w|} = \frac{\sqrt{k^2+1}}{k},$$

which finishes the proof. □

With the same technique and somewhat more effort one can give an embedding $\gamma_q$ of $T_{2,q}$ with $\delta(\gamma_q) \sim \frac{\pi}{2}\frac{q}{\log q}$. In addition a more technical proof yields that this asymptotical upper bound for $\delta(T_{2,q})$ is sharp for those embeddings of $T_{2,q}$ that project to a standard knot diagram via a linear projection. This let the author to the following.

**Question.** *Is $\delta(T_{2,q})$ up to a constant asymptotically equal to $q/\log q$? And if yes, is the constant equal to $\pi/2$?*

## References


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